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The Uniform over the Whole Line $\mathbb{R}$ Estimates of Spectral Expansions Related to the Selfadjoint Extensions of the Hill Operator and of the Schrödinger Operator with a Bounded and Measurable Potential

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Abstract—We consider some properties of the spectral expansions related to selfadjoint extensions of the operator $Hu = -u'' + q(x)u$ over the whole line $\mathbb{R}$ in the case when $q(x)$ is a continuous periodic function (the Hill operator) and in the case when $q(x)$ is a bounded measurable function. This paper gives a brief description of results obtained in the following directions: the uniform over $\mathbb{R}$ estimates of the generalized eigenfunctions, the uniform over $\mathbb{R}$ estimates of the spectral function, the uniform over $\mathbb{R}$ equiconvergence with the Fourier integral expansion, and the uniform over $\mathbb{R}$ rate of convergence for functions from the Sobolev-Liouville classes.

Keywords—Hill operator, One-dimensional Schrödinger operator, Spectral expansion.

The study of scattering on periodic structures demands the necessity for developing the spectral theory of the Hill operator over the whole line $\mathbb{R} = (-\infty, \infty)$ and, as its generalization, the spectral theory of the Schrödinger operator $Hu = -u'' + q(x)u$ with an arbitrary bounded and measurable over $\mathbb{R}$ potential $q(x)$.

In this paper, we review our results obtained in this direction. They relate mainly to the following problems:

(1) the problem of obtaining *the uniform over the whole line* $\mathbb{R}$ and sharp with respect to the order estimate for the integral of squares of generalized eigenfunctions of the selfadjoint extension of the Hill operator over $\mathbb{R}$;

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(2) the problem of obtaining *the uniform with respect to* $(x, y) \in \mathbb{R} \times \mathbb{R}$ estimate of the spectral function $\theta(\lambda;\ x, y)$ of the selfadjoint extension of the Hill operator over $\mathbb{R}$, and the uniform with respect to $x \in \mathbb{R}$ estimate of the $L_q(\mathbb{R})$-norm of $\theta(\lambda;\ x, y)$ in variable y $(2 \le q \le \infty)$;

(3) the problem of obtaining *the uniform over the whole line* $\mathbb{R}$ equiconvergence of the spectral expansion of an arbitrary function $f(x)$ from the class $L_p(\mathbb{R})$ $(1 \le p \le 2)$, related to the selfadjoint extension of the Hill operator over $\mathbb{R}$, with the conventional Fourier integral expansion of the same function $f(x)$;

(4) the problem of obtaining for an arbitrary function $f(x)$ from the Sobolev-Liouville class $L_2^\alpha(\mathbb{R})$ with $\alpha \in (1/2, 2]$ *the uniform over the whole line* $\mathbb{R}$ estimate of deviation of this function $f(x)$ from its spectral expansion related to the selfadjoint extension of the Hill operator over $\mathbb{R}$ or related to the Schrödinger operator with a bounded and measurable potential.

Note that there are a lot of papers and reviews which study different aspects of the spectral theory of the Hill operator (see, e.g., [1–7]). But only in [8,9], we managed to estimate the spectral function of the selfadjoint extension of the Hill operator, the related spectral expansion, and the deviation of this spectral expansion from the decomposed function in *the uniform over the whole line* $\mathbb{R}$ *metric*.

Let us describe and briefly analyse the obtained results.

It is known (see, e.g., [5, Chapter X.12]) that the Hill operator

$$Hu = -u'' + q(x) \cdot u, \tag{1}$$

with a continuous potential $q(x)$ satisfying periodic condition $q(x + a) = q(x)$, has the unique selfadjoint extension $\mathcal{L}$ over $\mathbb{R}$ that is bounded below. Since addition of an arbitrary constant to the potential $q(x)$ does not alter either its periodicity or the nature of the spectral expansion, but only shifts the spectrum, we assume without loss of generality that extension $\mathcal{L}$ is bounded below by 1.

It is also known (see [6, Chapter XIII.16]) that the spectral function $\theta(\lambda;\ x, y)$ of the selfadjoint extension $\mathcal{L}$ has the form

$$\theta(\lambda;\ x, y) = \frac{1}{\pi} \int_{-\infty}^{\lambda} \operatorname{Re}\left[\psi_t(x) \cdot \overline{\psi_t}(y)\right]\, dp(t), \tag{2}$$

where $p(t)$ is the so-called *quasi-momentum* (it is a continuous nondecreasing function over $\mathbb{R}$), and $\psi_\lambda(x)$ and $\overline{\psi_\lambda}(x)$ are *the Bloch eigenfunctions* that for any λ from the spectrum satisfy the equality $L\psi = \lambda\psi$ and can be obtained for a given λ as eigenfunctions of the monodromy operator $M(\lambda)$ with eigenvalues $\exp[\pm i\, p(\lambda) a]$.[1]

Furthermore, the spectrum of $\mathcal{L}$ is absolutely continuous and consists of an infinite number of bands:

$$[\lambda_0, \mu_0], \quad [\lambda_1, \mu_1], \qquad [\lambda_2, \mu_2], \dots.$$

The sequence $\lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots$ is formed by eigenvalues of the periodic (with period a) problem for the Hill operator (1), and the sequence $\mu_0 \le \mu_1 \le \mu_2 \le \cdots$ is formed by eigenvalues of the antiperiodic (with the same period a) problem for (1). The qualitative characterization of the quasimomentum $p(\lambda)$ is as follows: it equals zero on the initial gap $(-\infty, \lambda_0)$, equals $l\pi/a$

[1]The monodromy operator $M(\lambda)$ for a given λ is an operator which has (in the basis of solutions $c(x, \lambda)$ and $s(x, \lambda)$ of $Lu = \lambda u$, satisfying the initial conditions $c(0, \lambda) = s'(0, \lambda) = 1, s(0, \lambda) = c'(0, \lambda) = 0$) the following form:

$$\begin{pmatrix} c(a, \lambda) & s(a, \lambda) \\ c'(a, \lambda) & s'(a, \lambda) \end{pmatrix}.$$

The trace of this matrix $D(\lambda) = c(a, \lambda) + s'(a, \lambda)$ is called the Hill discriminant.

on the l^{th} gap (the numbering includes the gaps that degenerate into points), and satisfies the equation $2\cos[p(\lambda)a] = D(\lambda)$ on the bands (here $D(\lambda)$ is the Hill discriminant).

Taking into account the Browder-Gårding-Mautner Theorem on the ordered spectral representation of $L_2(\mathbb{R})$ space with respect to $\mathcal{L}$ (see, e.g., [10, Chapter XIII.5]) and using formula (2) for the spectral function $\theta(\lambda; x, y)$ of this extension, we conclude that the quasimomentum $p(\lambda)$ is the spectral measure of this spectral representation, and functions $u_1(x,\lambda)$ and $u_2(x,\lambda)$ which are defined on the bands by the equalities

$$u_1(x,t) = \frac{1}{\sqrt{\pi}} \operatorname{Re}[\psi_t(x)], \qquad u_2(x,t) = \frac{1}{\sqrt{\pi}} \operatorname{Im}[\psi_t(x)], \tag{3}$$

and arbitrarily on the gaps,[2] are the kernels (or the generalized eigenfunctions).

If the Fourier transforms of an arbitrary function $f(x)$ from $L_p(\mathbb{R})$, $1 \le p \le 2$ are defined as

$$\widehat{f}_i(\lambda) = \int_{-\infty}^{\infty} f(y)\, u_i(y,\lambda)\, dy, \qquad i = 1,2 \tag{4}$$

(in case $1 < p \le 2$, the integral (4) has meaning in the principal value sense), then the spectral expansion of $f(x)$ takes the form[3]

$$\sigma_\lambda(x,f) = \sum_{i=1}^{2} \int_{\lambda_0}^{\lambda} \widehat{f}_i(t)\, u_i(x,t)\, dp(t). \tag{5}$$

The Browder-Gårding-Mautner Theorem states that the spectral expansion (5) of an arbitrary function $f(x)$ in $L_2(\mathbb{R})$ converges to $f(x)$ in $L_2(\mathbb{R})$ metric as $\lambda \to \infty$, and the Parseval equality holds:

$$\sum_{i=1}^{2} \int_{\lambda_0}^{\infty} \widehat{f}_i^{\,2}(t)\, dp(t) = \int_{-\infty}^{\infty} f^2(x)\, dx. \tag{6}$$

The following statement forms the main result of [8].

MAIN THEOREM 1. *Let $p \in [1;2]$ be fixed. If $f(x)$ is an arbitrary function from $L_p(\mathbb{R})$, $\sigma_\lambda(x,f)$ is the spectral expansion of $f(x)$ related to the selfadjoint extension $\mathcal{L}$ of the Hill operator (1) over $\mathbb{R}$, $S_\lambda(x,f)$ is the Fourier integral expansion of $f(x)$, namely*

$$S_\lambda(x,f) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin\left[\sqrt{\lambda}(x-y)\right]}{x-y}\, f(y)\, dy, \tag{7}$$

then

$$\lim_{\lambda\to\infty} |\sigma_\lambda(x,f) - S_\lambda(x,f)| = 0,$$

uniformly with respect to x over the whole line $\mathbb{R}$.

This theorem yields that any convergence criterion of the Fourier integral expansion for a function in $L_p(\mathbb{R})$, $1 \le p \le 2$, at a point or on a set can serve as a convergence criterion (at a point or, respectively, on a set) of the spectral expansion related to the selfadjoint extension $\mathcal{L}$ of the Hill operator (1) over $\mathbb{R}$.

The proof of Main Theorem 1 essentially uses two auxiliary assertions which have a separate mathematical interest.

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[2] Functions $u_1(x,\lambda)$ and $u_2(x,\lambda)$ can be defined arbitrarily on the gaps because there we have $dp(\lambda) \equiv 0$. These functions only must satisfy $Lu_1(x,\lambda) = \lambda u_1(x,\lambda)$ and $Lu_2(x,\lambda) = \lambda u_2(x,\lambda)$.

[3] We take λ_0 instead of $-\infty$ as the lower bound in (5) because everywhere on the gap $(-\infty, \lambda_0) : dp(t) \equiv 0$. Note also that $\lambda_0 \ge 1$.

ASSERTION 1. *If $u_1(x,\lambda)$ and $u_2(x,\lambda)$ are the generalized eigenfunctions (3) of the ordered spectral representation of $L_2(\mathbb{R})$ with respect to the selfadjoint extension $\mathcal{L}$ of the Hill operator (1) over $\mathbb{R}$, then for all $\tau \geq \sqrt{\lambda_0} \geq 1$, the following estimate:*

$$\sum_{i=1}^{2} \int_{\tau \leq \sqrt{t} \leq \tau+1} |u_i(x,t)|^2 \, dp(t) = O(1), \tag{8}$$

holds uniformly with respect to $x \in \mathbb{R}$.

ASSERTION 2. *If q is an arbitrary fixed number satisfying $2 \leq q \leq \infty$, $\theta(\lambda;\, x,y)$ is the spectral function of the selfadjoint extension $\mathcal{L}$ of the Hill operator (1) over $\mathbb{R}$,*

$$\theta_0(\lambda;\, x,y) = \frac{1}{\pi} \frac{\sin\left[\sqrt{\lambda}(x-y)\right]}{x-y} \tag{9}$$

is the spectral function of the Fourier integral expansion, then the following uniform with respect to $x \in \mathbb{R}$ estimate:

$$\|\theta\,(\lambda; x, \cdot) - \theta_0\,(\lambda; x, \cdot)\|_{L_q(\mathbb{R})} = O(1), \tag{10}$$

holds for sufficiently large values of λ.

In [8], we also proved that the spectral function $\theta(\lambda;\, x,y)$ of the selfadjoint extension $\mathcal{L}$ of the Hill operator (1) over $\mathbb{R}$ satisfies the uniform with respect to $(x,y) \in \mathbb{R} \times \mathbb{R}$ estimate

$$|\theta(\lambda;\, x,y) - \theta_0(\lambda;\, x,y)| = O(1). \tag{11}$$

Estimates of the spectral function both for ordinary differential operators and for partial differential operators were considered by many well-known mathematicians (Carleman, Gårding, Levitan, Hörmander, Kostyuchenko, Avakumovich, etc.). Usually, in order to estimate the spectral function of a differential operator these authors applied the method initially introduced by Carleman. It is based on a preliminary estimate of some function of operator and on the subsequent application of Tauberian theorems.

Such a method gives estimates of the spectral function only in the case when both arguments x, y vary on compact sets of the domain where the given operator is considered.

Development of the method introduced in the monograph [11] allowed us to estimate in [8] the spectral function $\theta(\lambda;\, x,y)$ of the selfadjoint extension $\mathcal{L}$ of the Hill operator over $\mathbb{R}$ uniformly with respect to x, y varying independently over the infinite line $\mathbb{R}$.

Obtained estimates (10) and (11) of the spectral function give the necessary ground to prove [8], the uniform over the whole line $\mathbb{R}$ equiconvergence of the spectral expansion of an arbitrary function $f(x)$ from the class $L_p(\mathbb{R})$ $(1 \leq p \leq 2)$, which corresponds to the selfadjoint extension $\mathcal{L}$ of the Hill operator over $\mathbb{R}$, with the conventional Fourier integral expansion of the same function $f(x)$.

While using spectral expansions in various applications, it is important to know estimates for the deviation of the decomposed function $f(x)$ from its spectral expansion $\sigma_\lambda(x,f)$ as $\lambda \to \infty$ (provided we know which class the function $f(x)$ is from). Furthermore it is important to obtain these estimates in the uniform with respect to $x \in \mathbb{R}$ metric.

This problem is covered in [9]. In that paper, we consider any bounded below selfadjoint extension $\mathcal{L}$ of the Schrödinger operator

$$Hu = -u'' + q(x)u, \qquad x \in \mathbb{R}, \tag{12}$$

with an arbitrary bounded and measurable over $\mathbb{R}$ potential $q(x)$.

Some particular cases of this Schrödinger operator are of a special importance:

(1) the operator (12) with a continuous periodic potential $q(x)$ (the Hill operator),
(2) the operator (12) with the potential $q(x) = \beta\cos(\gamma x)$, where β and γ are constants (the Mathieu-Hill operator),
(3) the operator (12) with an almost periodic potential $q(x)$.

Without loss of generality, we assume this extension $\mathcal{L}$ to be strictly positive with spectrum's infimum $\lambda_0 \geq 1$. Indeed, the general case of a selfadjoint bounded below extension $\mathcal{L}$ can be reduced to it if we add a constant to the potential $q(x)$ (after this transformation, the new potential is also bounded and measurable, while the whole spectrum shifts).

The following statement forms the main result of [9].

MAIN THEOREM 2. *Let $\mathcal{L}$ be a selfadjoint and bounded below by $\lambda_0 \geq 1$ extension of the Schrödinger operator (12) over $\mathbb{R}$ with the potential $q(x)$, which is an arbitrary bounded and measurable over $\mathbb{R}$ function. Let $f(x)$ be any function from the Sobolev-Liouville class $L_2^\alpha(\mathbb{R})$, where $\alpha \in (1/2, 2]$, and $\sigma_\lambda(x, f)$ be the spectral expansion of $f(x)$ related to $\mathcal{L}$. Then the difference $\sigma_\lambda(x, f) - f(x)$ satisfies the following uniform for $x \in \mathbb{R}$ estimate:*

$$|\sigma_\lambda(x, f) - f(x)| = \lambda^{1/4-\alpha/2} \cdot o(1); \tag{13}$$

here $o(1)$ denotes an entity that tends to zero as $\lambda \to \infty$ uniformly over $\mathbb{R}$.

The estimate (13) clarifies the uniform over the whole line $\mathbb{R}$ rate of vanishing of $|\sigma_\lambda(x, f) - f(x)|$ as $\lambda \to \infty$.

The proof of Main Theorem 2 uses one auxiliary result that expresses the main characteristic property of the Fourier transforms of a function $f(x)$ from the Sobolev-Liouville class $L_2^\alpha(\mathbb{R})$. This result also has a separate interest.

THEOREM 3. *Let $\mathcal{L}$ be a selfadjoint bounded below by $\lambda_0 \geq 1$ extension of the Schrödinger operator (12) over $\mathbb{R}$ with the potential $q(x)$, which is an arbitrary bounded and measurable over $\mathbb{R}$ function. Let $f(x)$ be any function from the Sobolev-Liouville class $L_2^\alpha(\mathbb{R})$ with a fixed derivation order $\alpha \in (0, 2]$. Then the Fourier transforms $\widehat{f}_i(t)$ of $f(x)$ related to the ordered spectral representation[4] of $L_2(\mathbb{R})$ space with respect to the extension $\mathcal{L}$ satisfy the following inequality:*

$$\sum_{i=1}^{m} \int_{\lambda_0}^{\infty} \left|\widehat{f}_i(t)\right|^2 \cdot t^\alpha \, dp(t) \leq C \|f\|^2_{L_2^\alpha(\mathbb{R})}. \tag{14}$$

Following the well-known Bernstein's Theorem on absolute and uniform convergence of the Fourier trigonometric series for a periodic function $f(x)$ from the Hölder class $C^\alpha[-\pi, \pi]$ with $\alpha > 1/2$ (see, e.g., [12, Chapter 6, Theorem 3.1]), it is natural to introduce the notion of the absolute and uniform over the whole line $\mathbb{R}$ convergence of the spectral expansion

$$\sigma_\lambda(x, f) = \sum_{i=1}^{m} \int_{\lambda_0}^{\lambda} \widehat{f}_i(t) \, u_i(x, t) \, dp(t) \tag{15}$$

of an arbitrary function $f(x)$.

We say that the spectral expansion (15) *converges absolutely and uniformly over the whole line* $\mathbb{R}$, if the integral

$$\int_{\lambda_0}^{\infty} \left|\widehat{f}_i(t)\right| \cdot |u_i(x, t)| \, dp(t)$$

converges for all $x \in \mathbb{R}$ and $i = \overline{1, m}$, and there exists the uniform with respect to $x \in \mathbb{R}$ limit

$$\lim_{\lambda \to \infty} \sum_{i=1}^{m} \int_{\lambda}^{\infty} \left|\widehat{f}_i(t)\right| \cdot |u_i(x, t)| \, dp(t) = 0. \tag{16}$$

[4]This representation is characterized by the spectral measure $p(t)$, multiplicity $m \leq 2$, multiplicity sets e_i, and the generalized eigenfunctions $u_i(x, t)$.

THEOREM 4 (GENERALIZED THEOREM OF BERNSTEIN'S TYPE). *If $\mathcal{L}$ is a selfadjoint bounded below by $\lambda_0 \geq 1$ extension of the Schrödinger operator (12) over $\mathbb{R}$ with a bounded and measurable over $\mathbb{R}$ potential $q(x)$, then the corresponding spectral expansion (15) of an arbitrary function $f(x)$ from the Sobolev-Liouville class $L_2^\alpha(\mathbb{R})$ with $\alpha > 1/2$ converges absolutely and uniformly over the whole line $\mathbb{R}$.*

Theorem 4 is also proved in [9]. Note that as in all other theorems of [8,9], the absolute and uniform convergence of the spectral expansion (15) is established *over the whole line* $\mathbb{R}$.

In [9], we also prove that in the important case of the Mathieu-Hill operator

$$Hu = -u'' + \beta\cos(\gamma x)u, \qquad x \in \mathbb{R}, \tag{17}$$

the restriction $\alpha \leq 2$ in Main Theorem 2 and in Theorem 3 can be removed. For the Mathieu-Hill operator (17) the following statement holds.

THEOREM 5. *Let $\mathcal{L}$ be a selfadjoint bounded below by $\lambda_0 \geq 1$ extension of the Mathieu-Hill operator (17) over $\mathbb{R}$, and $f(x)$ be any function from the Sobolev-Liouville class $L_2^\alpha(\mathbb{R})$. Then the estimate (14) for the Fourier transforms $\widehat{f}_i(t)$ of $f(x)$ holds for any fixed $\alpha \in (0,\infty)$, and the uniform over the whole line $\mathbb{R}$ estimate[5] (13) for the difference of the function $f(x)$ and its spectral expansion $\sigma_\lambda(x,f)$ holds for any fixed $\alpha \in (1/2,\infty)$.*

Thus in the important case of the Schrödinger operator, the Mathieu-Hill operator (17), one can get the desired rate of vanishing of $|\sigma_\lambda(x,f) - f(x)|$ as $\lambda \to \infty$ (uniformly with respect to $x \in \mathbb{R}$), choosing the decomposed function $f(x)$ from the Sobolev-Liouville class with a sufficiently high derivation order α.

Concluding, we note that analysis of the proof of Main Theorem 1 (see [8]) and the estimate for the integral of squares of generalized eigenfunctions (see [13]) makes it possible to state that Main Theorem 1 holds true, also in the case when the Hill operator is substituted by the Schrödinger operator with a bounded and measurable potential $q(x)$.

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[5]See Main Theorem 2 for the explanation of the notation $o(1)$ in (13).